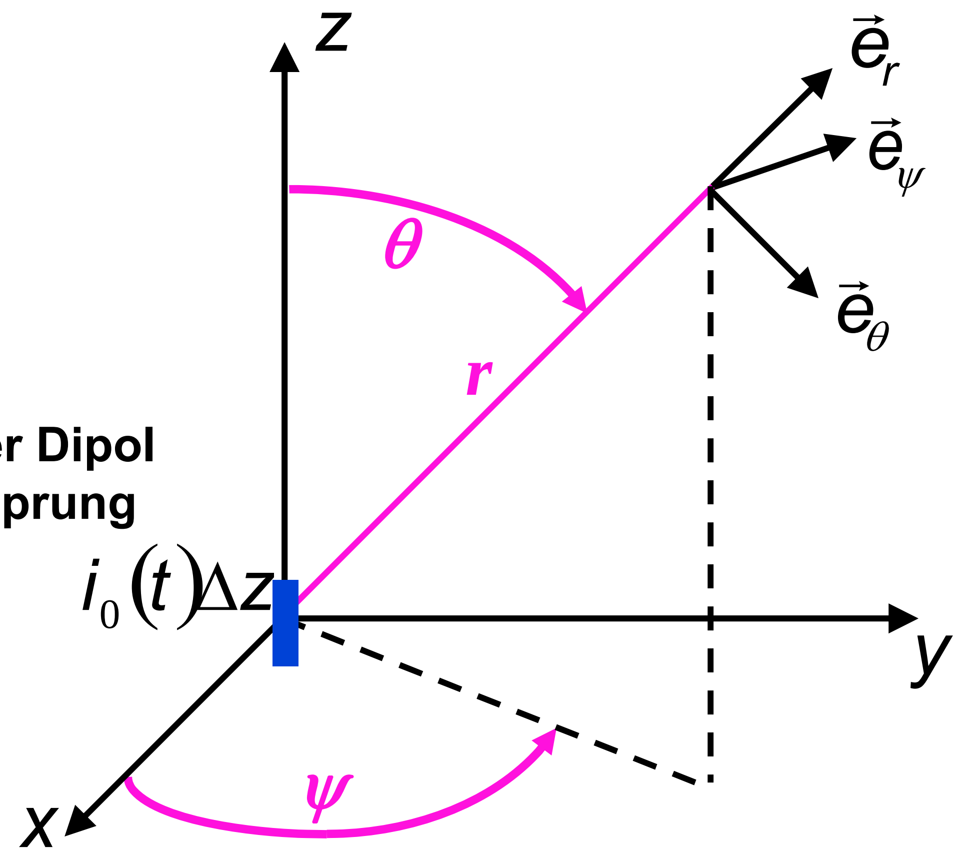
Der Hertz ‘sche Dipol

by Thomas Zwick

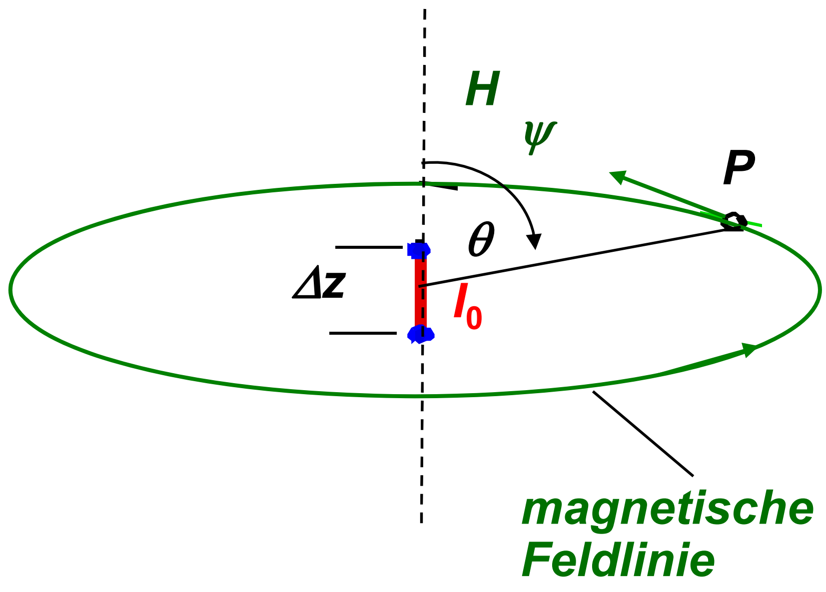
INSTITUT FÜR HOCHFREQUENZTECHNIK UND ELEKTRONIK

Hertz´scher Dipol im Kugelkoordinatensystem

Infinitesimal kleiner Dipol
im Koordinatenursprung



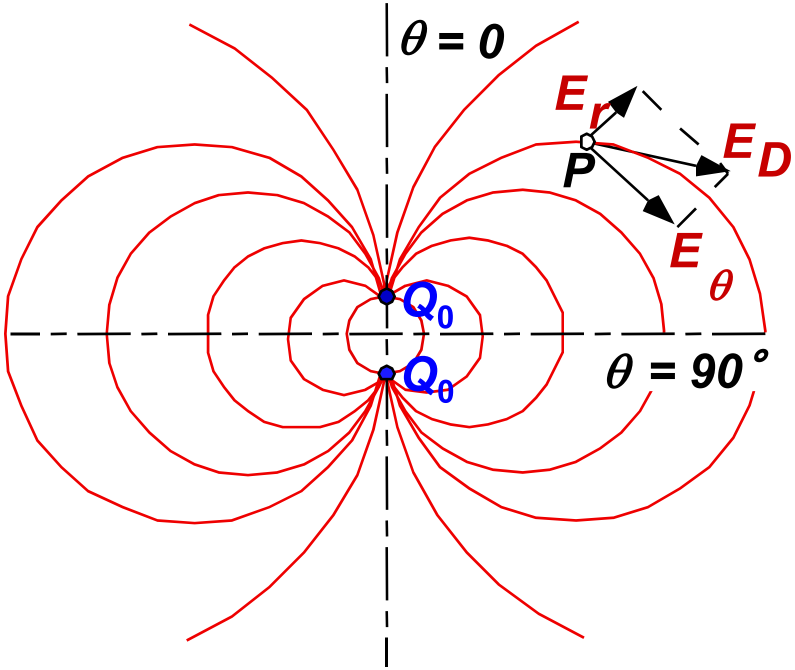
Magnetisches Feld eines stromdurchflossenen Leiters



Magnetfeld eines infinitesimalen, Strom durch-flossenen Leiters nach Gesetz von Biot-Savart:

$$d\vec{H} = \frac{I_0}{4\pi} \frac{d\vec{z} \times \vec{e}_r}{\left|\vec{r}\right|^2}$$

Elektrisches Feld zweier Ladungen

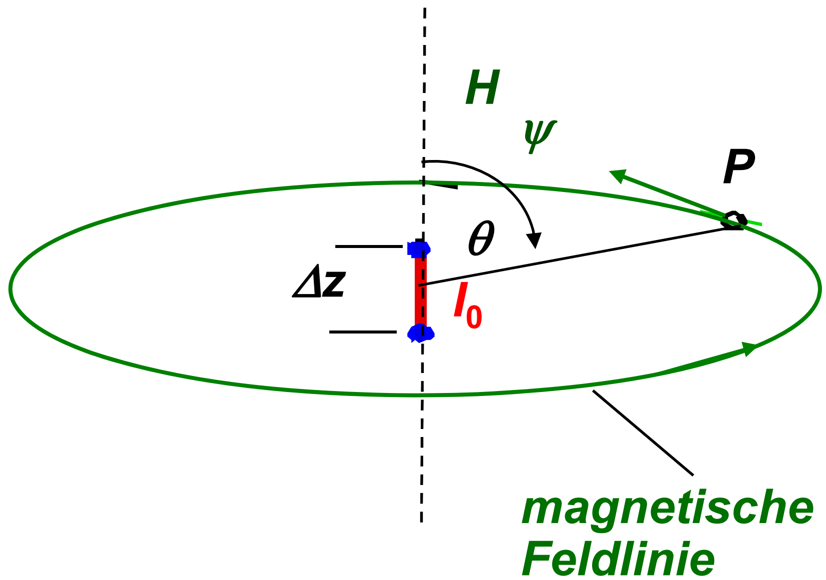


Elektrisches Feld eines elektrostatischen Dipols nach Coulomb-Gesetz:

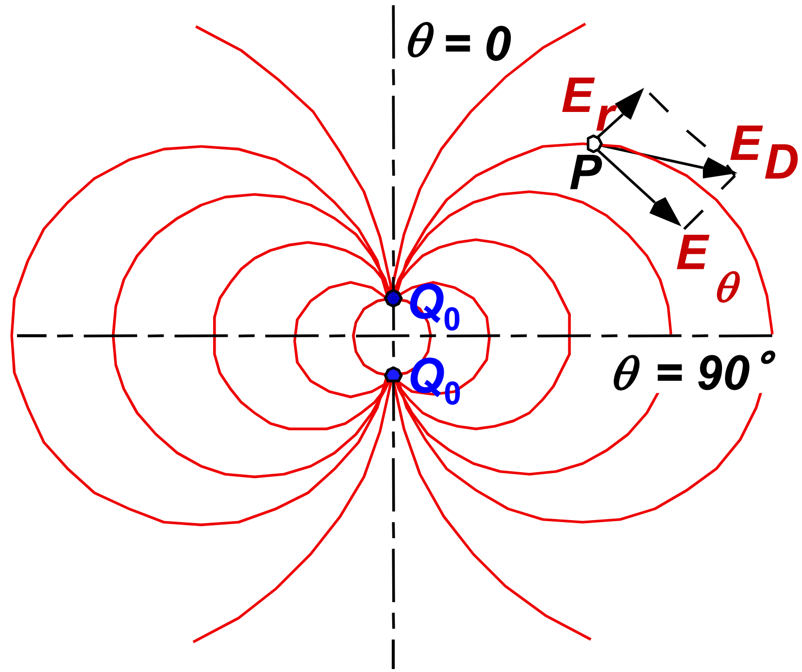
$$E_r = \frac{Q_0 \Delta z \cos\theta}{2\pi\varepsilon r^3} \qquad E_\theta = \frac{Q_0 \Delta z \sin\theta}{4\pi\varepsilon r^3}$$

Berechnung des Nahfelds aus statischen Feldern

Feld eines stromdurchflossenen Leiters



Feld zweier Ladungen



nach Gesetz von Biot-Savart:

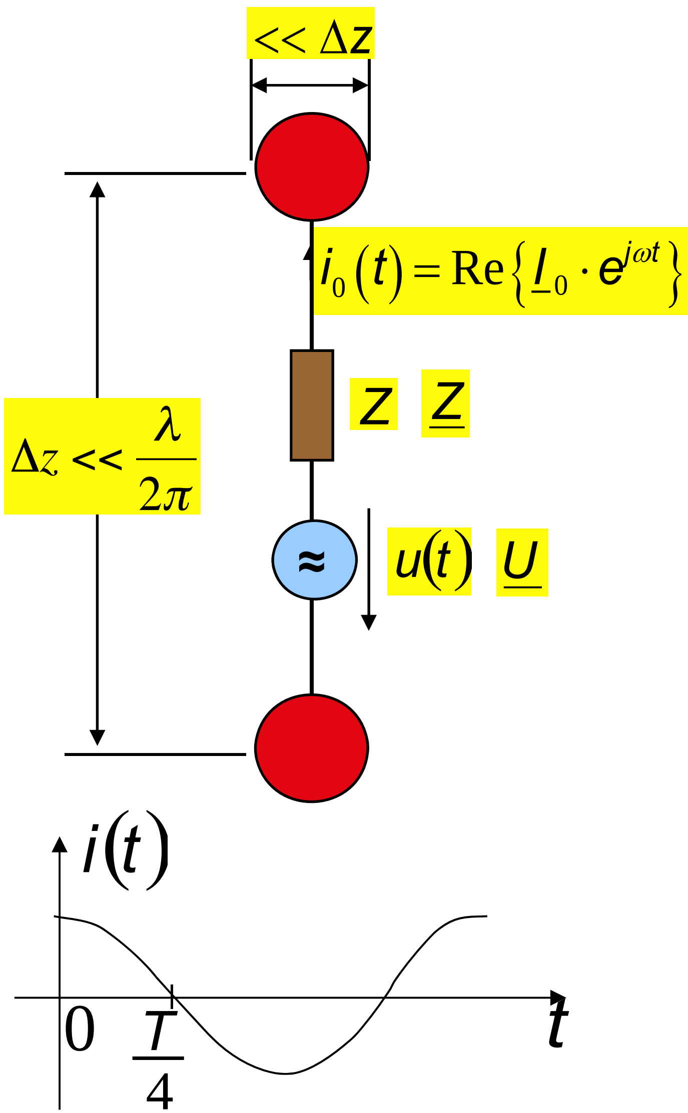
$$H_\psi = \frac{I_0 \Delta z \sin\theta}{4\pi r^2}$$

nach Coulomb-Gesetz:

$$E_r = -j\frac{I_0 \Delta z \cos\theta}{2\omega\pi\varepsilon r^3} \qquad E_\theta = -j\frac{I_0 \Delta z \sin\theta}{4\omega\pi\varepsilon r^3}$$

E und H um 90 Grad phasen-verschoben! -> keine Abstrahlung!

Berechnung aus Maxwell Gleichungen



Inhomogene Wellengleichung:

$$\vec{\nabla}^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

Punktquelle & Strom nur in z-Richtung:

$$J_z = 0 \quad \text{für} \quad r > 0$$

$$\vec{J} = J_z \cdot \vec{e}_z \quad \text{für} \quad r = 0$$

$$\Rightarrow \quad \vec{A} = A_z \cdot \vec{e}_z$$

$$\frac{\partial A_z}{\partial \theta} = 0 \quad \& \quad \frac{\partial A_z}{\partial \psi} = 0$$

$$\Rightarrow \quad A_z = A_z(r)$$

$$\vec{\nabla}^2 A_z(r) + k^2 A_z(r) = -\mu J_z$$

Lösung der homogenen Wellengleichung

Homogenen Wellengleichung:

$$\vec{\nabla}^2 A_z(r) + k^2 A_z(r) = 0$$

$$\vec{A} = A_z(r) \cdot \vec{e}_z$$

$$\frac{\partial A_z}{\partial \theta} = 0 \quad \& \quad \frac{\partial A_z}{\partial \psi} = 0$$

In Kugelkoordinaten:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z}{\partial r}\right) + k^2 A_z(r) = 0 \qquad A_z = \frac{\Omega}{r} \quad \Rightarrow \quad \frac{\partial A_z}{\partial r} = \frac{\partial \Omega}{r \partial r} - \frac{\Omega}{r^2}$$

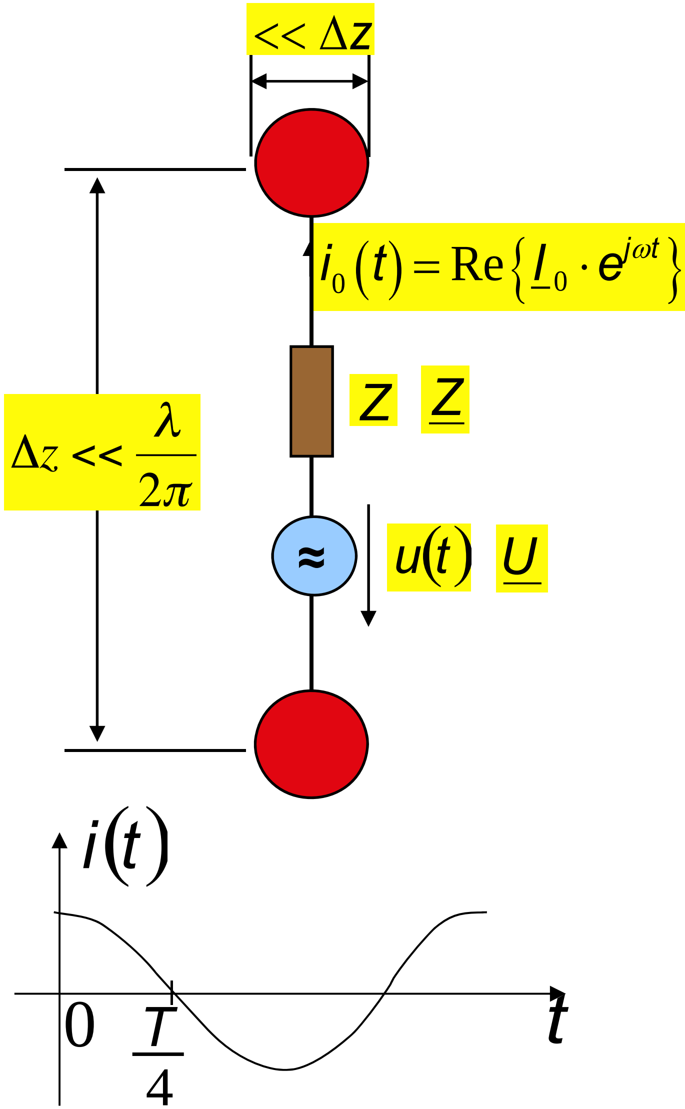
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r \frac{\partial \Omega}{\partial r} - \Omega\right) + k^2 \frac{\Omega}{r} = \frac{1}{r^2}\left(\frac{\partial \Omega}{\partial r} + r\frac{\partial^2 \Omega}{\partial r^2} - \frac{\partial \Omega}{\partial r}\right) + k^2 \frac{\Omega}{r} = 0$$

$$\frac{\partial^2 \Omega}{\partial r^2} + k^2 \Omega = 0 \quad \Rightarrow \quad \Omega = C_{1,2} e^{\pm jkr} \quad \Rightarrow \quad A_z = A_0 \frac{e^{-jkr}}{r}$$

Formelsammlung: Räumliche Differentiation in Kugelkoordinaten

$$\vec{\nabla}^2 \varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \varphi}{\partial \psi}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \varphi}{\partial \psi^2}$$

Berechnung aus Maxwell Gleichungen



Inhomogene Wellengleichung:

$$\vec{\nabla}^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

Punktquelle & Strom nur in z-Richtung:

$$J_z = 0 \quad \text{für} \quad r > 0$$

$$\vec{J} = J_z \cdot \vec{e}_z \quad \text{für} \quad r = 0$$

$$\Rightarrow \quad \vec{A} = A_z \cdot \vec{e}_z$$

$$\frac{\partial A_z}{\partial \theta} = 0 \quad \& \quad \frac{\partial A_z}{\partial \psi} = 0$$

$$\Rightarrow \quad A_z = A_z(r)$$

$$A_z = A_0 \frac{e^{-jkr}}{r} \quad \text{mit} \quad A_0 = \mathrm{func}(\underline{I}_0)$$

Institut für Hochfrequenztechnik und Elektronik

Lösung der inhomogenen Wellengleichung (1)

für kleine Sphäre mit Radius $r_0 \to 0$ (Integration der inh. WG)

$$\int_V \vec{\nabla}^2 A_z \, dV = -k^2 \int_V A_z \, dV - \mu \int_V J_z \, dV$$

Linke Seite:

Satz von Gauss: $\iiint \operatorname{div} \vec{u} \, dv = \oiint \vec{u} \, d\vec{a}$

$$\Rightarrow \quad \int_V \vec{\nabla} \cdot \vec{\nabla} A_z \, dV = \int_0^{2\pi} \int_0^{\pi} \vec{\nabla} A_z \vec{e}_r r^2 \sin\theta \, d\theta \, d\psi$$

$$mit \quad \vec{\nabla} A_z \vec{e}_r = \frac{\partial A_z}{\partial r} = A_0 \left(-\frac{e^{-jkr}}{r^2} - jk \frac{e^{-jkr}}{r} \right)$$

$$\lim_{r_0 \to 0} \int_V \vec{\nabla} \cdot \vec{\nabla} A_z \, dV = \lim_{r_0 \to 0} \int_0^{2\pi} \int_0^{\pi} A_0 \left(-1 - jkr_0 \right) e^{-jkr_0} \sin\theta \, d\theta \, d\psi = -4\pi A_0$$

Formelsammlung: Räumliche Differentiation in Kugelkoordinaten

$$\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin\theta} \frac{\partial \varphi}{\partial \psi} \vec{e}_\psi$$

Institut für Hochfrequenztechnik und Elektronik

Lösung der inhomogenen Wellengleichung (2)

für kleine Sphäre mit Radius $r_0 \rightarrow 0$ (Integration der inh. WG)

$$\int_V \vec{\nabla}^2 A_z \, dV = -k^2 \int_V A_z \, dV - \mu \int_V J_z \, dV$$

Rechte Seite:

$$\lim_{r_0 \to 0} \int_V A_z dV = \lim_{r_0 \to 0} \int_0^{r_0} \int_0^{2\pi} \int_0^{\pi} A_0 \frac{e^{-jkr}}{r} r^2 \sin\theta \, d\theta \, d\psi \, dr =$$

$$\lim_{r_0 \to 0} \int_0^{r_0} A_0 r e^{-jkr} 4\pi \, dr = 4\pi A_0 \lim_{r_0 \to 0} \left[\frac{r}{-jk} e^{-jkr} + \frac{1}{k^2} e^{-jkr} \right]_0^{r_0} = 0$$

$$\lim_{r_0 \to 0} \int_V J_z dV = I_0 d\ell \qquad da \qquad dV = da \cdot d\ell$$

Gesamt:

$$-4\pi A_0 = -\mu I_0 d\ell \quad \Rightarrow \quad A_0 = \frac{\mu I_0 d\ell}{4\pi} \quad \Rightarrow \quad \vec{A} = \mu I_0 d\ell \frac{e^{-jkr}}{4\pi r} \vec{e}_z$$

Allgemeine Lösung der Wellengleichung

$$k = 0 \quad \Rightarrow \quad A_z = A_0 \frac{1}{r} \quad \& \quad \vec{\nabla}^2 A_z = -\mu J_z \quad \text{(Poisson - Gleichung)}$$

$$A_z = \frac{\mu}{4\pi} \iiint\limits_V \frac{J_z}{r} dV'$$

$$k > 0 \quad \Rightarrow \quad A_z = \frac{\mu}{4\pi} \iiint\limits_V \frac{J_z e^{-jkr}}{r} dV'$$

$$A_z(r,t) = \frac{\mu}{4\pi} \iiint\limits_V \frac{J_z e^{-jkr} e^{j\omega t}}{r} dV'$$

$$\frac{k}{\omega} = \frac{2\pi}{\lambda 2\pi f} = \frac{1}{c} \quad \Rightarrow \quad e^{-jkr} e^{j\omega t} = e^{j\omega\left(t - \frac{k}{\omega} r\right)}$$

$$\vec{A}(\vec{r},t) = \frac{\mu}{4\pi} \iiint\limits_V \frac{J_z\left(\vec{r}\,', t - \frac{1}{c}\left|\vec{r} - \vec{r}\,'\right|\right)}{\left|\vec{r} - \vec{r}\,'\right|} dV'$$

Institut für Hochfrequenztechnik
und Elektronik

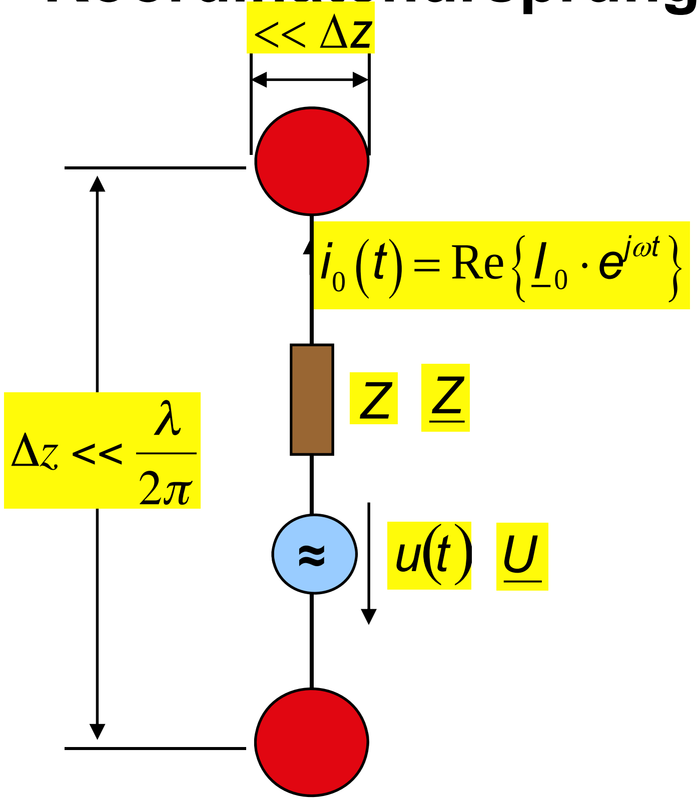
Wellengleichung mit allgemeiner Lösung

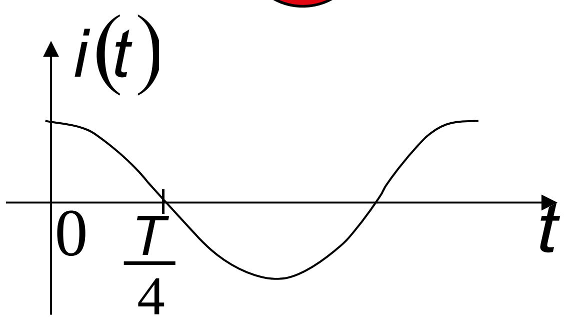
Karlsruhe Institute of Technology

Inhomogene Wellengleichung: $\vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$

Allgemeine Lösung: $\vec{A}(\vec{r},t) = \frac{\mu}{4\pi} \iiint_V \frac{\vec{J}\left(\vec{r}\,',t - \frac{1}{c}\left|\vec{r} - \vec{r}\,'\right|\right)}{\left|\vec{r} - \vec{r}\,'\right|} dV'$

Infinitesimal kleiner Dipol im Koordinatenursprung





$$\vec{J}(\vec{r}\,',t)dV' = \vec{J}(\vec{r}\,',t)da'dz' = \begin{cases} 0 & \text{für } \vec{r}\,' \neq 0 \\ i_0(t)\vec{e}_z dz' & \text{für } \vec{r}\,' = 0 \end{cases}$$

für r>0

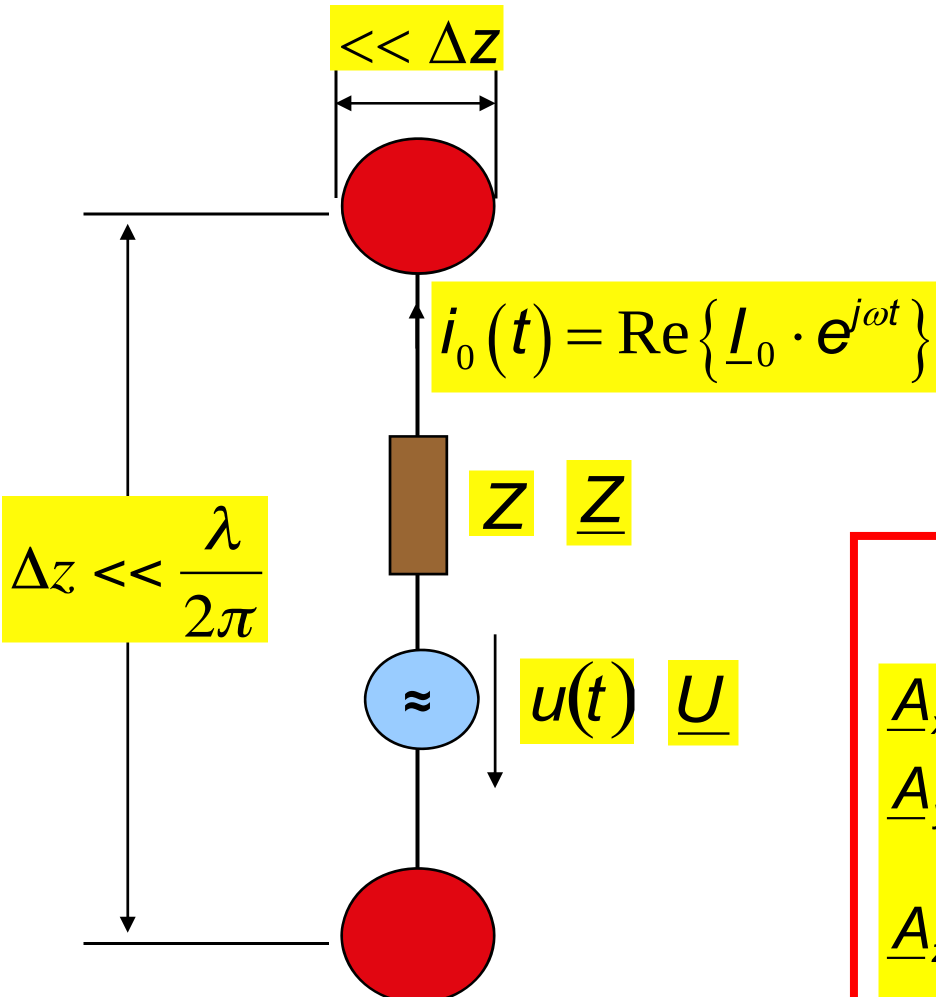
$$\vec{A}(\vec{r},t) = \frac{\mu}{4\pi} \frac{i_0\left(t - \frac{1}{c}|\vec{r}|\right)}{|\vec{r}|} \Delta z \cdot \vec{e}_z$$

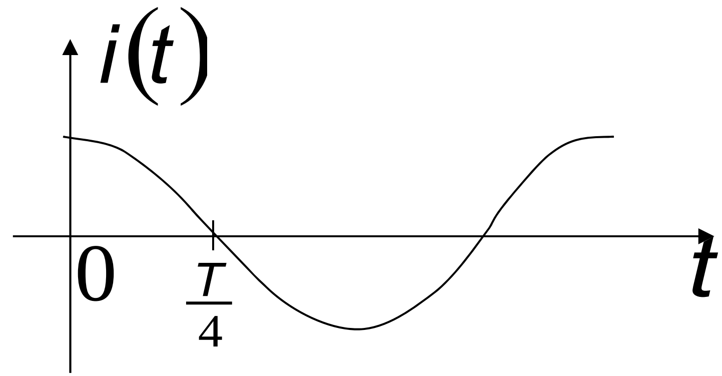
harmonische Vorgänge / verlustloses Medium

$$\vec{A}(r,t) = \mathrm{Re}\left\{ \frac{\mu \underline{I}_0 \Delta z}{4\pi r} e^{j\omega\left(t - \frac{r}{c}\right)} \right\} \vec{e}_z \qquad r = |\vec{r}|$$

$$\Rightarrow \quad \underline{A}_0 = \frac{\mu \underline{I}_0 \Delta z}{4\pi}$$

Infinitesimal kleiner Dipol im Koordinatenursprung





$$\underline{A}_x = 0$$

$$\underline{A}_y = 0$$

$$\underline{A}_z = \underline{A}_0 \frac{e^{-jkr}}{r}$$

$$\underline{\vec{H}} = \frac{1}{\mu} \mathrm{rot}\, \underline{\vec{A}}$$

$$\underline{\vec{J}} = 0 \quad \text{für } r > 0$$

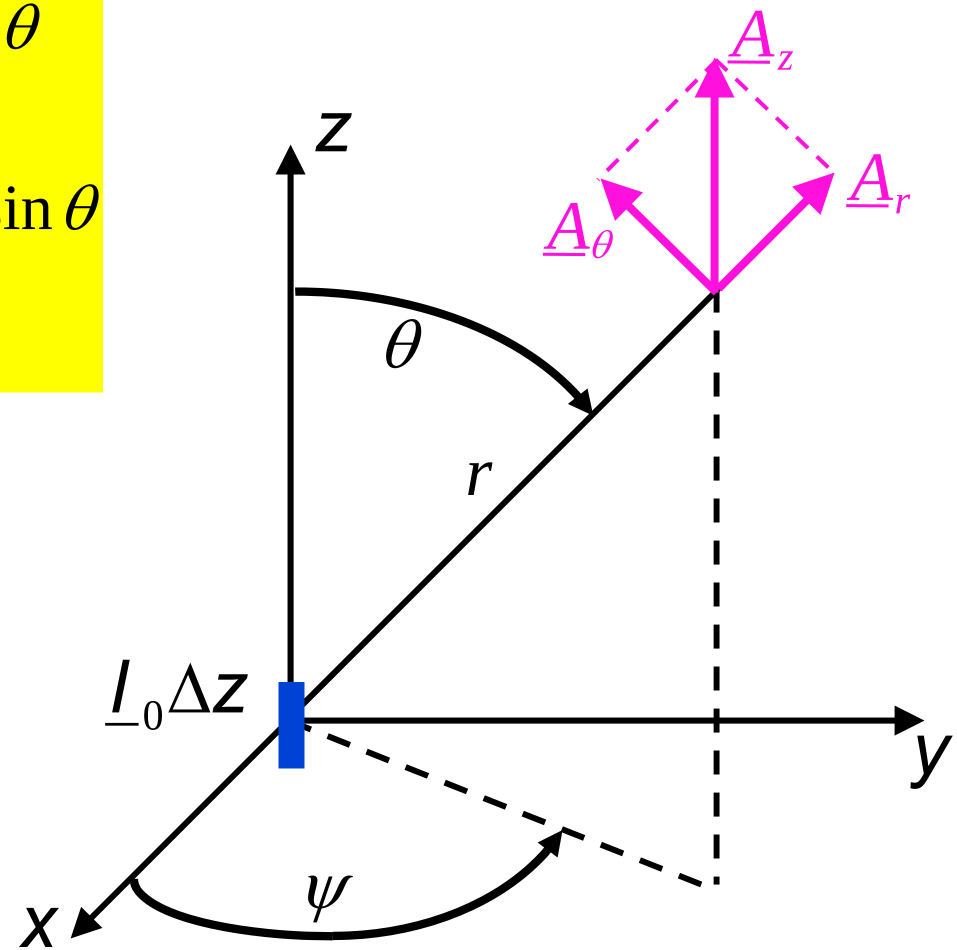
$$\Rightarrow \underline{\vec{E}} = \frac{1}{j\omega\varepsilon} \mathrm{rot}\, \underline{\vec{H}}$$

Koordinatensystem für Vektorpotenzial

$$\underline{A}_r = \underline{A}_z \cos\theta = \underline{A}_0 \frac{e^{-jkr}}{r} \cos\theta$$

$$\underline{A}_\theta = -\underline{A}_z \sin\theta = -\underline{A}_0 \frac{e^{-jkr}}{r} \sin\theta$$

$$\underline{A}_\psi = 0$$



Bestimmung der Felder des Hertz´schen Dipols

$$\underline{A}_r = \underline{A}_0 \frac{e^{-jkr}}{r}\cos\theta$$

$$\underline{A}_\theta = -\underline{A}_0 \frac{e^{-jkr}}{r}\sin\theta$$

$$\underline{A}_\psi = 0$$

$$\underline{\vec{H}} = \frac{1}{\mu}\,\mathrm{rot}\,\underline{\vec{A}}$$

H-Feld für r>0

$$\underline{H}_r = 0$$

$$\underline{H}_\theta = 0$$

$$\underline{H}_\psi = \frac{jk^2\,\underline{I}_0\Delta z\sin\theta}{4\pi}\left(\frac{1}{kr} - \frac{j}{(kr)^2}\right)e^{-jkr}$$

$$\underline{\vec{E}} = \frac{1}{j\omega\varepsilon}\,\mathrm{rot}\,\underline{\vec{H}}$$

E-Feld für r>0

$$\underline{E}_r = -\frac{jZ_F k^2\,\underline{I}_0\Delta z\cos\theta}{2\pi}\left(\frac{j}{(kr)^2} + \frac{1}{(kr)^3}\right)e^{-jkr}$$

$$\underline{E}_\theta = \frac{jZ_F k^2\,\underline{I}_0\Delta z\sin\theta}{4\pi}\left(\frac{1}{kr} - \frac{j}{(kr)^2} - \frac{1}{(kr)^3}\right)e^{-jkr}$$

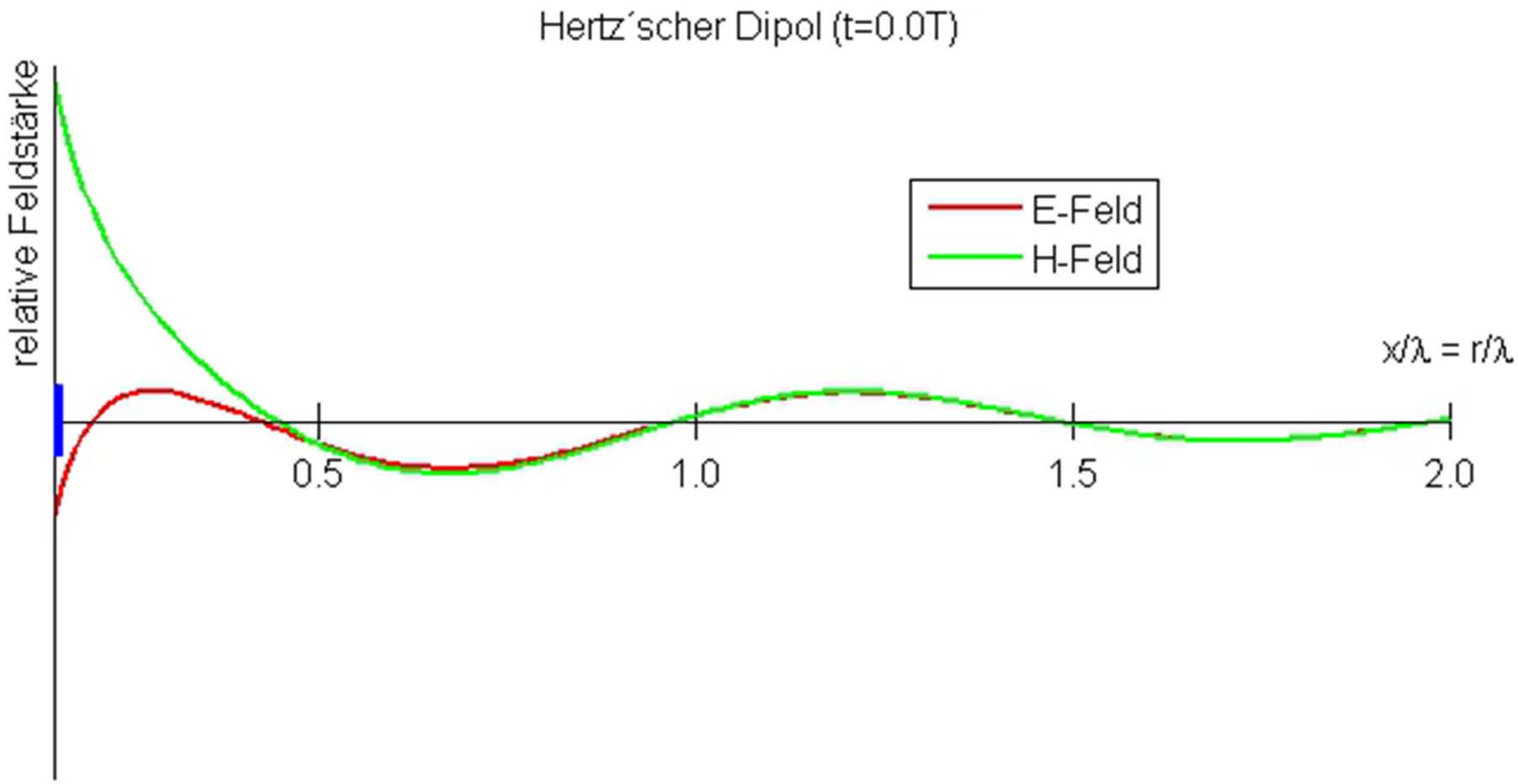
$$\underline{E}_\psi = 0$$

Relative Feldstärke abhängig vom Abstand

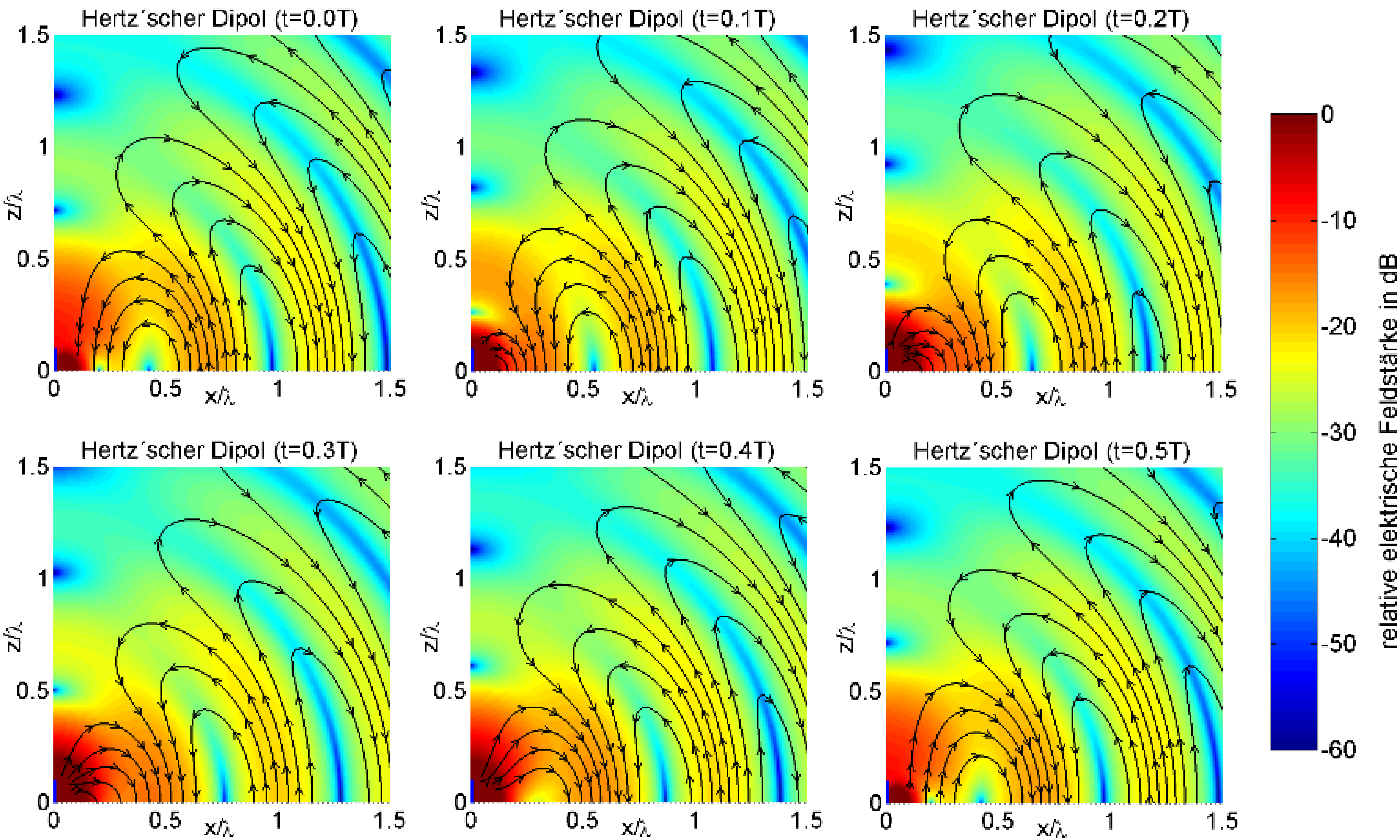
$$\mathrm{Re}\left\{ e^{j\omega t}\sqrt{\left|\underline{E}_r\right|^2+\left|\underline{E}_\theta\right|^2}\right\}$$

$$\mathrm{Re}\left\{ e^{j\omega t}\left|\underline{H}_\psi\right|\right\}$$

$$T=\frac{1}{f}$$

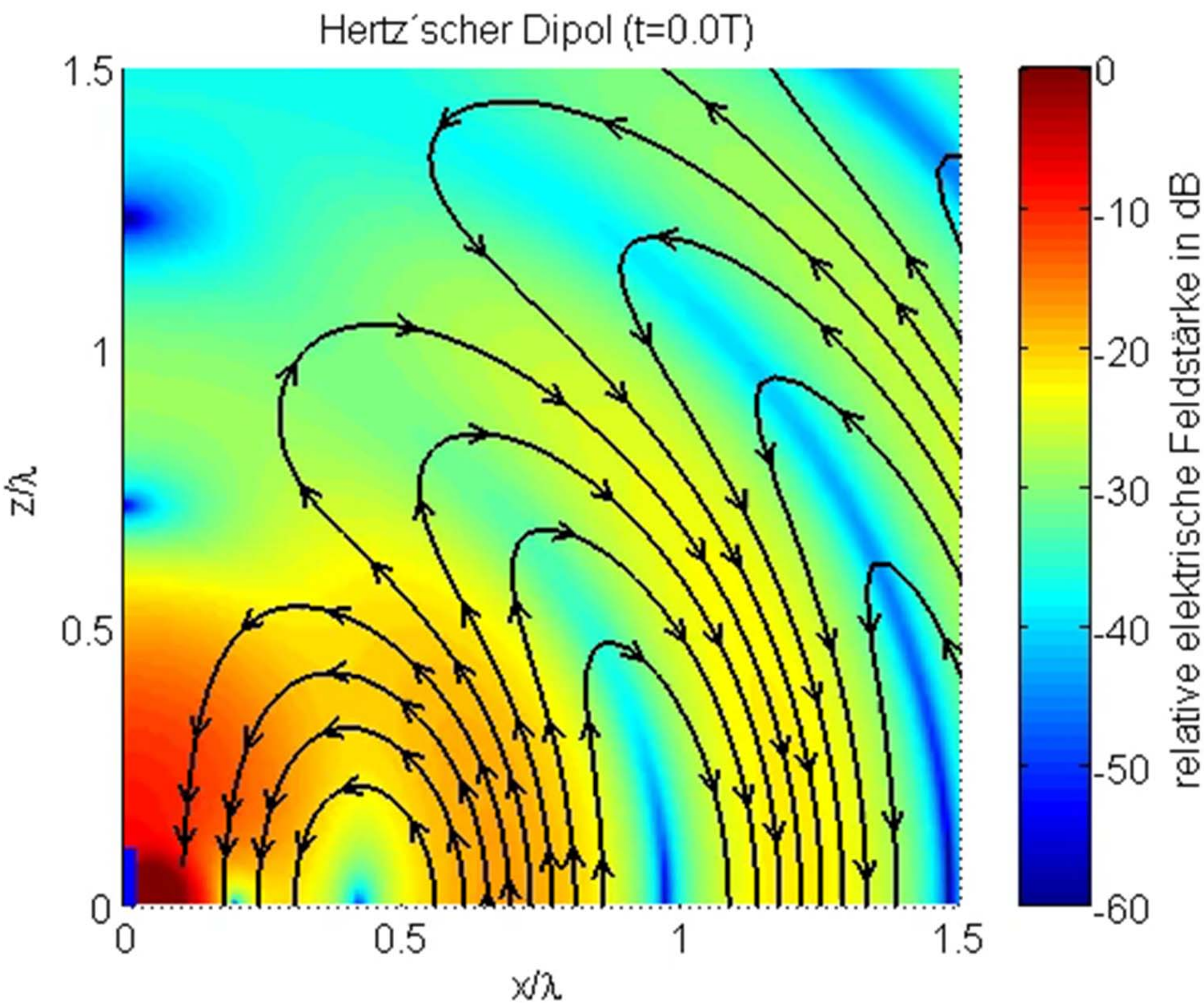


Elektrisches Feld eines Hertz´schen Dipols in verschiedenen Phasen (*T* = *1/f* = Periodendauer)



Institut für Hochfrequenztechnik
und Elektronik

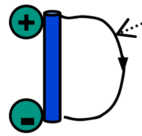
Fernfeld des Hertz´schen Dipols



Wellenablösung am Dipol

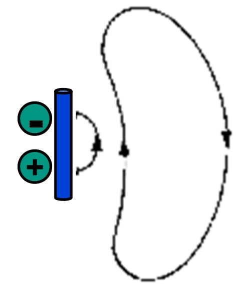
E-Feld

$t = 0$

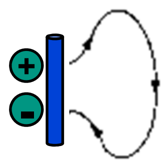


I=0

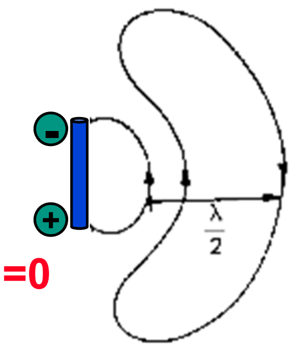
$t = \frac{3T}{8}$



$t = \frac{T}{8}$

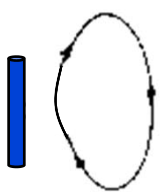


$t = \frac{T}{2}$



I=0

$t = \frac{T}{4}$



I=max

$$\left.\begin{array}{l} T = 1/f \\ c = \lambda \cdot f \end{array}\right\} \Rightarrow \lambda = c \cdot T$$

Felder des Hertz´schen Dipols

H-Feld für r>0

$$\underline{H}_r = 0$$

$$\underline{H}_\theta = 0$$

$$\underline{H}_\psi = \frac{jk^2 \underline{I}_0 \Delta z \sin\theta}{4\pi}\left(\frac{1}{kr} - \frac{j}{(kr)^2}\right)e^{-jkr}$$

Nahfeld des Hertz ʻschen Dipols:
$kr << 1$, $e^{-jkr} \approx 1$

Fernfeld des Hertz ʻschen Dipols:
$kr >> 1$

E-Feld für r>0

$$\underline{E}_r = -\frac{jZ_F k^2 \underline{I}_0 \Delta z \cos\theta}{2\pi}\left(\frac{j}{(kr)^2} + \frac{1}{(kr)^3}\right)e^{-jkr}$$

$$\underline{E}_\theta = \frac{jZ_F k^2 \underline{I}_0 \Delta z \sin\theta}{4\pi}\left(\frac{1}{kr} - \frac{j}{(kr)^2} - \frac{1}{(kr)^3}\right)e^{-jkr}$$

$$\underline{E}_\psi = 0$$

Institut für Hochfrequenztechnik und Elektronik

Nahfeld des Hertz'schen Dipols: $kr \ll 1,\ e^{-jkr} \approx 1$

$$\underline{E}_r = -\frac{jZ_F \underline{I}_0 \Delta z \cdot \cos\theta}{2\pi \cdot kr^3}$$

$$\underline{E}_\theta = -\frac{jZ_F \underline{I}_0 \Delta z \cdot \sin\theta}{4\pi \cdot kr^3}$$

$$\underline{E}_\psi = 0$$

$$\underline{H}_r = 0$$

$$\underline{H}_\theta = 0$$

$$\underline{H}_\psi = \frac{\underline{I}_0 \Delta z \cdot \sin\theta}{4\pi \cdot r^2}$$

$$\underline{\vec{S}} = \frac{1}{2}\underline{\vec{E}} \times \underline{\vec{H}}^*$$

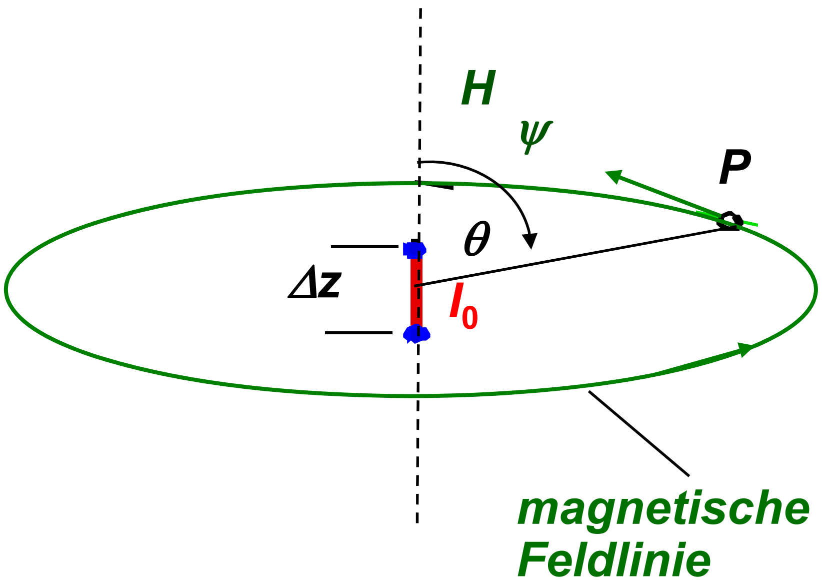
Der Pointing Vektor ist imaginär:

⇨ **Energiespeicherung**

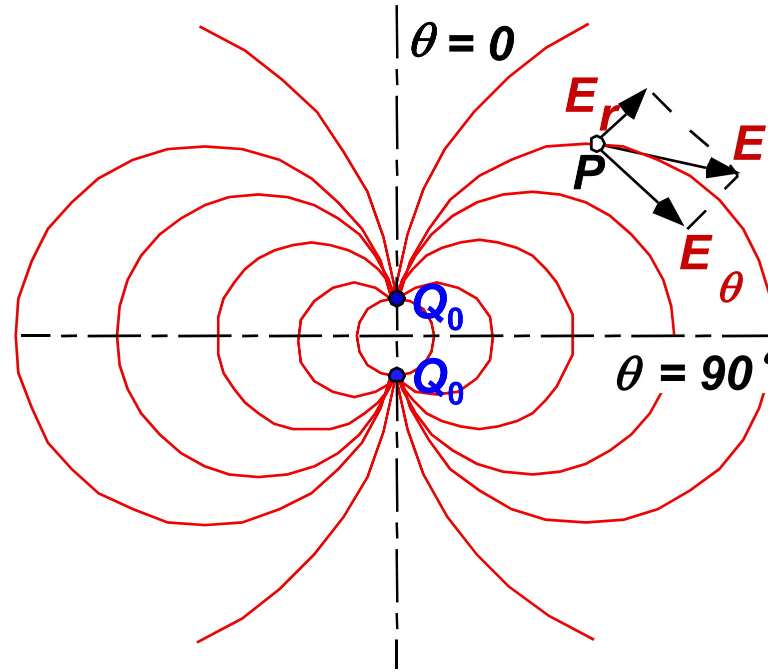
⇨ **Blindleistung**

Berechnung des Nahfelds aus statischen Feldern

Feld eines stromdurchflossenen Leiters



Feld zweier Ladungen



nach Gesetz von Biot-Savart:

$$H_\psi = \frac{I_0 \Delta z \sin\theta}{4\pi r^2}$$

nach Coulomb-Gesetz:

$$E_r = -j\frac{I_0 \Delta z \cos\theta}{2\omega\pi\varepsilon r^3} \qquad E_\theta = -j\frac{I_0 \Delta z \sin\theta}{4\omega\pi\varepsilon r^3}$$

Fernfeld des Hertz ‘schen Dipols: *kr >> 1*

$$\underline{E}_r = 0$$

$$\underline{E}_\theta = \frac{jZ_F k \underline{I}_0 \Delta z \cdot \sin\theta}{4\pi \cdot r} e^{-jkr}$$

$$\underline{E}_\psi = 0$$

$$\underline{H}_r = 0$$

$$\underline{H}_\theta = 0$$

$$\underline{H}_\psi = \frac{jk \underline{I}_0 \Delta z \cdot \sin\theta}{4\pi \cdot r} e^{-jkr}$$

$$\frac{\underline{E}_\theta}{\underline{H}_\psi} = Z_F \qquad S_r = \frac{1}{2} \underline{E}_\theta \cdot \underline{H}_\psi^* = \frac{Z_F \cdot k^2 \left(\left|\underline{I}_0\right| \Delta z\right)^2}{32\pi^2 r^2} \sin^2\theta$$

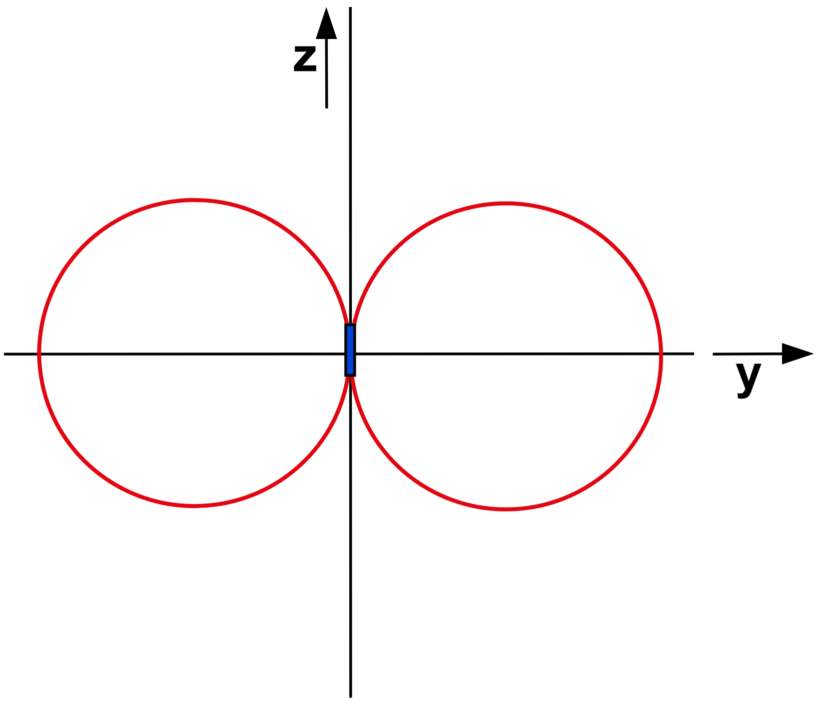
- **Fernfeld hat nur transversale Komponenten**
- **Der Pointing Vektor ist reell und zeigt in Ausbreitungsrichtung**

$\Longrightarrow$ **WIRKLEISTUNGSTRANSPORT!**

Richtdiagramm des Hertz´schen Dipols (2D)

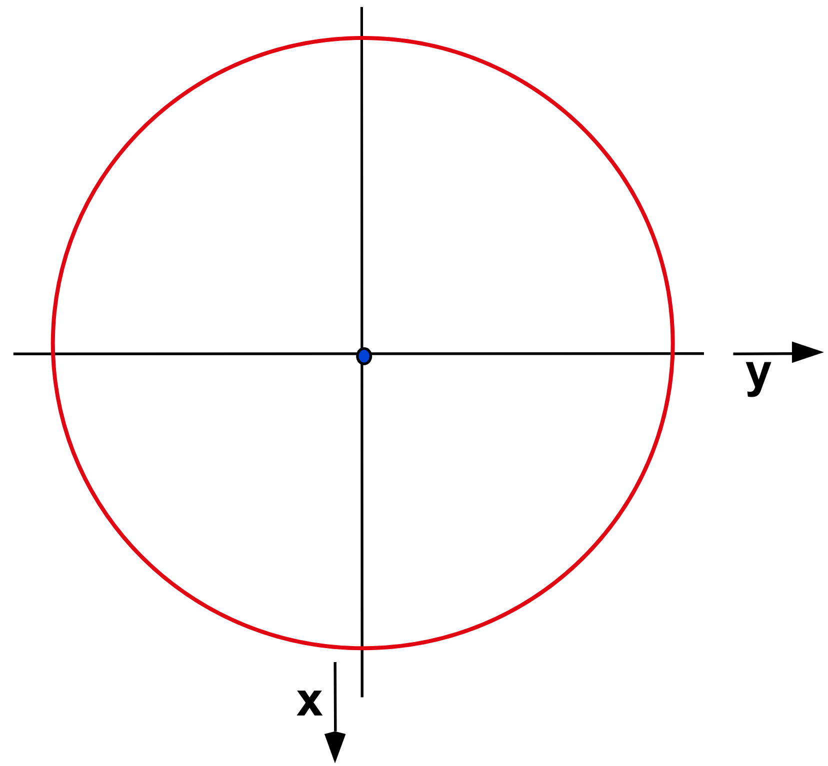
Vertikaldiagramm

(Seitenansicht)



Horizontaldiagramm

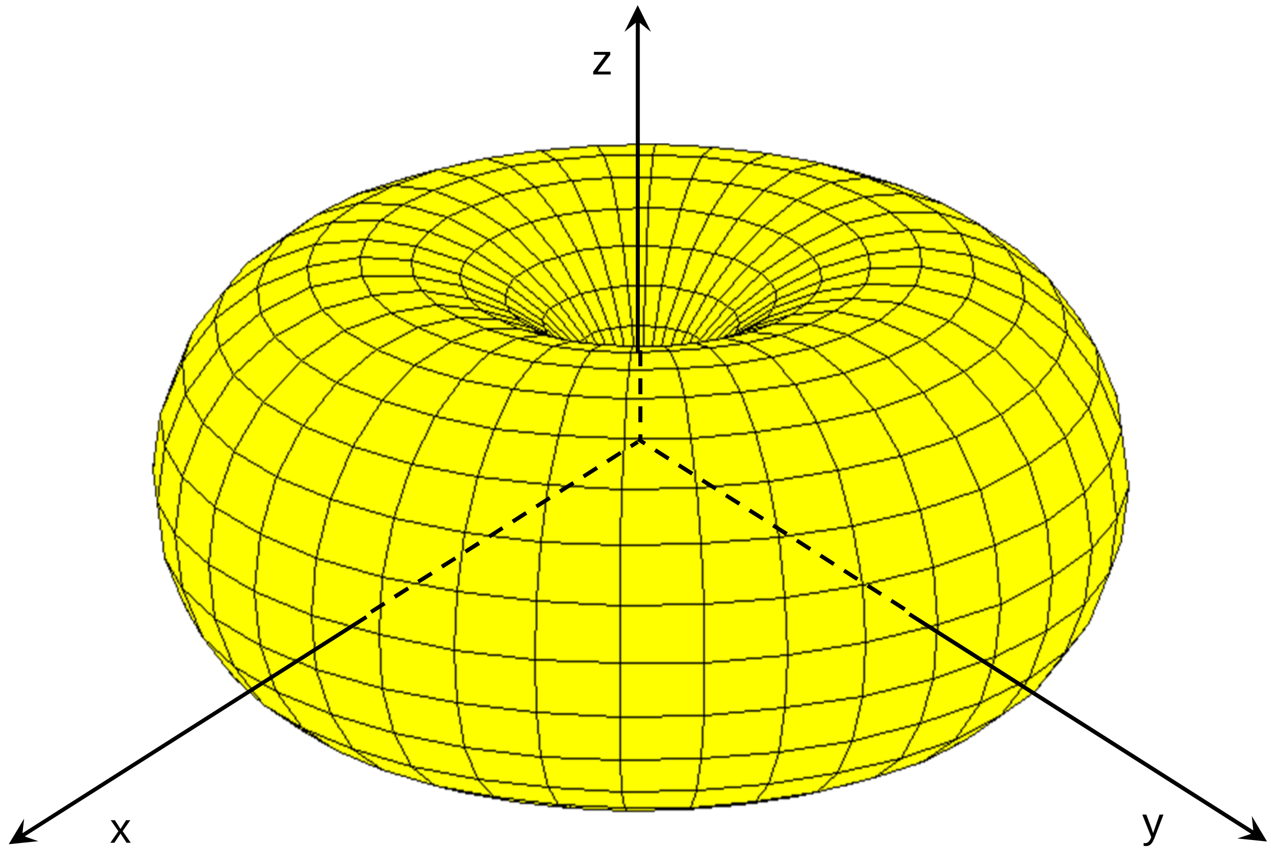
(Draufsicht)



$$C(\theta,\psi) = \sin\theta$$

Institut für Hochfrequenztechnik
und Elektronik

Räumliche Richtcharakteristik des Hertz´schen Dipols

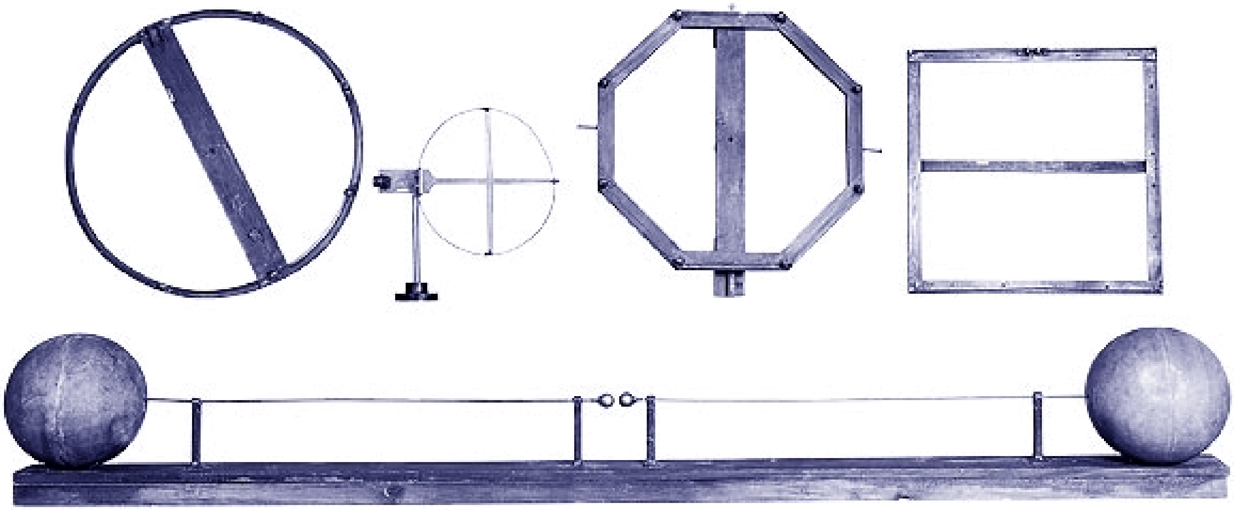


$$C(\theta,\psi) = \sin\theta$$

Heinrich Hertz

* 22.02.1857
† 01.01.1894

Experimente zum Nachweis von elektromagnetischen Wellen in den Jahren 1886 bis 1889 in Karlsruhe



Eigenschaften Hertz ‘scher Dipol (1)

$$P_T = r^2 \int_{\psi=0}^{2\pi} \int_{\theta=0}^{\pi} S_r \sin\theta \, d\theta \, d\psi \qquad S_r = \frac{1}{2} E_\theta H_\psi^* = \frac{1}{8} |I_0|^2 Z_{F0} \left(\frac{\Delta z}{\lambda_0}\right)^2 \frac{\sin^2\theta}{r^2}$$

$$P_T = \frac{1}{8} |I_0|^2 Z_{F0} \left(\frac{\Delta z}{\lambda_0}\right)^2 \int_{\psi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin^3\theta \, d\theta \, d\psi =$$

$$\frac{1}{8} |I_0|^2 Z_{F0} \left(\frac{\Delta z}{\lambda_0}\right)^2 2\pi \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_{\theta=0}^{\pi} = \frac{\pi}{3} |I_0|^2 Z_{F0} \left(\frac{\Delta z}{\lambda_0}\right)^2$$

$$\text{Strahlungswiderstand:} \quad P_T = \frac{1}{2} |I_0|^2 R_S$$

$$R_S = \frac{2\pi}{3} Z_{F0} \left(\frac{\Delta z}{\lambda_0}\right)^2 = 790\Omega \left(\frac{\Delta z}{\lambda_0}\right)^2 \quad \text{für} \quad \frac{\Delta z}{\lambda_0} << \frac{1}{2\pi}$$

$$G_i = 4\pi r^2 \frac{S_{r,\max}}{P_T} = 4\pi r^2 \frac{\frac{1}{8} |I_0|^2 Z_{F0} \left(\frac{\Delta z}{\lambda_0}\right)^2 \frac{1}{r^2}}{\frac{\pi}{3} |I_0|^2 Z_{F0} \left(\frac{\Delta z}{\lambda_0}\right)^2} = \frac{3}{2} \quad (1.76 dBi)$$

Eigenschaften Hertz ‘scher Dipol (2)

$$\text{Ebene Welle}: \; E = E_{x0} e_x e^{jkz} \qquad H = H_{y0} e_y e^{jkz}$$

$$S = S_z e_z = \frac{1}{2} E_{x0} H_{y0}^* e_z \qquad S_z = \frac{1}{2Z_{F0}} \left| E_{x0} \right|^2$$

$$U_A = E_{x0} \Delta z \qquad \text{(infinitesimal kleiner Hertz'scher Dipol)}$$

$$P_R = A_w S_z = A_w \frac{\left| E_{x0} \right|^2}{2Z_{F0}}$$

Maximal verfügbare Leistung (konjugiert komplexe Anpassung):

$$P_R = \frac{\left| U_A \right|^2}{8R_S} \qquad (U_A \text{ ist Scheitelwertgröße!})$$

$$A_w = \frac{\left| U_A \right|^2}{8R_S} \frac{2Z_{F0}}{\left| E_{x0} \right|^2} = \frac{Z_{F0} (\Delta z)^2}{4R_S} = \frac{3}{8\pi} \lambda_0^2 \quad \Rightarrow \quad A_w = \frac{\lambda_0^2}{4\pi} G_i$$

Institut für Hochfrequenztechnik und Elektronik